PROJECTED WRITTEN NOTES FROM THE M 325K LECTURE ON THURSDAY, APRIL 18, 2024, ON SECTION 7.4 - CARDINALITY OF SETS AND "NOTES on CARDINALITY"

(T, Th) CLASS # 26

We begin with a review of THEOREM (NIB) 10, which is found in the handout called "MORE ON ONE-TO-ONE FUNCTIONS and Onto Functions and One-to-one and Onto Functions."

Theorem (NIB) 10: Let $f : X \to Y$ and $g : Y \to X$ be functions.

If $g \circ f = i_X$ and $f \circ g = i_Y$,

then f is a one-to-one correspondence and $g = f^{-1}$.

Proof: Recall: $i_X(x) = x, \forall\ x \in X$, and $i_Y(y) = y, \forall\ y \in Y$.

Since i_X is one-to-one and since f is applied first in $g \circ f$,
f is one-to-one by Part 1) of Theorem (NIB) 4.
Since i_Y is onto and since f is applied last in $f \circ g$,
f is onto Part 2) of Theorem (NIB) 4.
$\therefore$ f is a one-to-one correspondence.

The proof that $g = f^{-1}$ is left as an exercise. (#25 of Sec. 7.3) Q E D

We continued with an example of applying Theorem (NIB) 10 found on the second page of this same handout.

"The Test for a One-to-one Correspondence":

Theorem (NIB) 10 can be used to prove that a function f is a one-to-one correspondence by taking the following steps:

Step 1) Compute what the formula for $g = f^{-1}$ should be.

Step 2) Show that $g \circ f = i_X$ and $f \circ g = i_Y$; that is, show that $g(f(x)) = x$, for all $x \in X$, and that $f(g(y)) = y$, for all $y \in Y$.

Example Problem:

Using the "Test for a One-to-one Correspondence", prove that f is a one-to-one correspondence from $\mathbb{R}$ to $\mathbb{R}^+$, where:

For all $x \in \mathbb{R}$, $f(x) = 5e^{3x} \in \mathbb{R}^+ = Y$

$x \in X$

Solution: Recall that, $\forall x \in \mathbb{R}$, $e^{\ln x} = x$, and $\forall y \in \mathbb{R}^+$, $\ln\left(e^{y}\right) = y$.

Step 1) Compute what the formula for $g = f^{-1}$ should be:

$$g(y) = x \Leftrightarrow f^{-1}(y) = x \Leftrightarrow f(x) = y \Leftrightarrow 5e^{3x} = y \Leftrightarrow e^{3x} = y/5$$

$$\Leftrightarrow 3x = \ln\left(\frac{y}{5}\right) \Leftrightarrow x = \frac{1}{3}\ln\left(\frac{y}{5}\right)$$

Define function $g: \mathbb{R}^+ \to \mathbb{R}$ as follows: For all $y \in \mathbb{R}^+$, $g(y) = \frac{1}{3}\ln\left(\frac{y}{5}\right)$.

Step 2) Show that $g \circ f = i_X$ and $f \circ g = i_Y$; that is, show that $g(f(x)) = x$, for all $x \in X$, and that $f(g(y)) = y$, for all $y \in Y$.

$$f\circ g(y) = f(g(y)) = f\left(\frac{1}{3}\ln\left(\frac{y}{5}\right)\right) = 5e^{\left(3\left(\frac{1}{3}\ln\left(\frac{y}{5}\right)\right)\right)} = 5e^{\left(\ln\left(\frac{y}{5}\right)\right)} = 5\frac{y}{5} = y = I_Y(y)$$

$$g\circ f(x) = g(f(x)) = g\left(5e^{3x}\right) = \frac{1}{3}\ln\left(\frac{5e^{3x}}{5}\right) = \frac{1}{3}\ln\left(e^{3x}\right) = \frac{1}{3}3x = x = I_X(x)$$

Therefore, f is a one-to-one correspondence and $g = f^{-1}$ by Theorem (NIB) 10, since $g \circ f = I_X$ and $f \circ g = I_Y$.

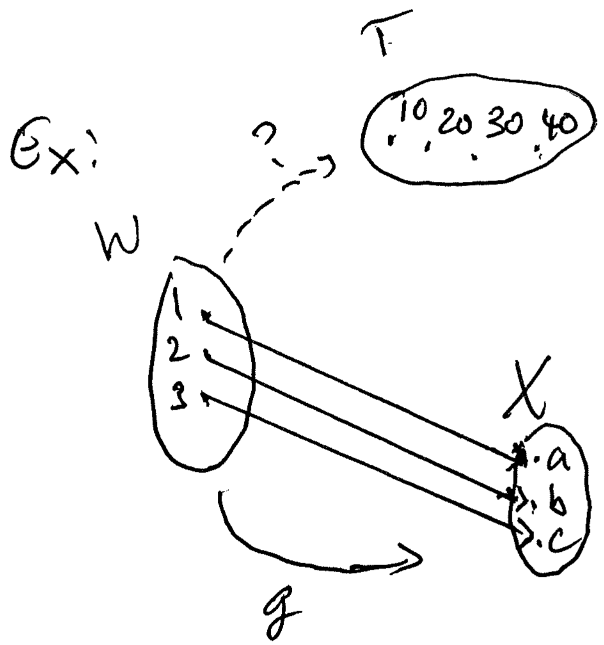
CARDINALITY OF SETS

Def'n: Two Sets A and B "have the SAME CARDINALITY" (Write: CARD(A) = CARD(B))

$\Longleftrightarrow$

There exists a one-to-one correspondence $f: A \to B$ [In which case, $f^{-1}: B \to A$ is also a one-to-one correspondence]

Ex:



No function $f: W \to T$ can be a one-to-one correspondence.

CARD(W) $\neq$ CARD(T)

Since g is a one-to-one correspondence, "CARD(W) = CARD(X)", i.e. "W and X have the same Cardinality."

For finite sets, CARD(A) = CARD(B) $\Rightarrow$ A and B have the same # of elements.

Def'n: A set X is said to be "countably infinite"

$\Longleftrightarrow$

$$\text{CARD}(X) = \text{CARD}(\mathbb{Z}^+)$$

Def'n: A set is finite $\Longleftrightarrow$

① $A = \emptyset$

OR

② $\text{CARD}(A) = \text{CARD}(\{1, 2, 3, \ldots, n\})$ for some integer $n \geq 1$.

Def'n: A set X is a countable set

$\Longleftrightarrow$ X is finite or X is countably infinite.

Def'n: A set Y is uncountable

$\Longleftrightarrow$ Y is not finite AND NOT countably infinite.

Notes on Cardinality

Definitions: Let A and B be non-empty sets.

A has the same cardinality as B if, and only if,
there exists a one-to-one correspondence from A to B, that is,

there exists a function $f: A \to B$ such that f is a one-to-one correspondence, that is,

there exists a function $f: A \to B$ such that f is one-to-one and onto.

As discussed below,
B has the same cardinality as A if, and only if, A has the same cardinality as B.

Thus, it is unambiguous to use the phrase "A and B have the same cardinality" to mean both, that B has the same cardinality as A and A has the same cardinality as B.

Observations: (Theorem 7.4.1)

A) For all sets A, A has the same cardinality as A. (Reflexive Property)

Proof: Using i_A, the identity function of set A, $i_A : A \to A$ is a one-to-one correspondence from A to A. Thus, A has the same cardinality as A.

B) For all sets A and B,
if A has the same cardinality as B, then B has the same cardinality as A.
(Symmetric Property)

Proof: Since A has the same cardinality as B, there exists a function $f: A \to B$ such that f is a one-to-one correspondence. Its inverse function $f^{-1} : B \to A$ is a one-to-one correspondence from B to A. Thus, B has the same cardinality as A.

C) For all sets A, B, and C,
if A has the same cardinality as B and B has the same cardinality as C,
then A has the same cardinality as C. (Transitive Property)

Proof: Since A has the same cardinality as B and B has the same cardinality as C, there exist functions $f: A \to B$ and $g: B \to C$ such that f and g are both one-to-one correspondences. Therefore, the composition $g \circ f: A \to C$ is a one-to-one correspondence. Thus, A has the same cardinality as C.

2)

Definitions:

A non-empty set B is finite if, and only if, there exists a positive integer n and a function f: $\{ 1, 2, 3, \ldots, n \} \to B$ such that f is a one-to-one correspondence.

In this case, we say that n is the "number of elements in set B;" we say "the cardinality of the set B is n;" and we say "B has the same cardinality as $\{ 1, 2, 3, \ldots, n \}$."

A non-empty set is Infinite if, and only if, it is not finite.

A non-empty set B is Countably Infinite
if, and only if, B has the same cardinality as $\mathbb{Z}^+$, that is,
there exists a function f: $\mathbb{Z}^+ \to B$ such that f is a one-to-one correspondence.

A non-empty set B is Countable
if, and only if, B is finite or countably infinite.

A set B is Uncountable if and only if B is not countable, that is,
if and only if B is infinite and not countably infinite.

The empty set $\varnothing$ will be considered to be a finite set.

Theorem (NIB) 11:

Given any non-empty countable set A, it is possible to list all of the elements of A in a sequence $a_1, a_2, a_3, \ldots, a_n$ (in the case that A is finite) or in a or in a sequence $a_1, a_2, a_3, \ldots$ (in the case that A is infinite) such that every element of A appears in the sequence once and only once.

Thus, for any countable set A, such a listing of the elements of A is possible.

Proof: Suppose A is any non-empty countable set

Thus, A is finite or A is infinite.

Case 1 (A is finite):

Suppose A is a finite set.

Then, by definition the of "finite set", there exists a positive integer n and a function f: $\{ 1, 2, 3, \ldots, n \} \to A$ such that f is a one-to-one correspondence.

Define $a_1 = f(1)$, $a_2 = f(2)$, $a_3 = f(3), \ldots, a_n = f(n)$.

Then, the sequence $a_1, a_2, a_3, \ldots, a_n$ has no repetitions (since f is one-to-one) and the sequence includes all the elements of A (since f is onto).

Thus, such a listing of the elements of A is possible in the case that A is a finite set.

SHOWING THAT $\mathbb{Z}^+$, the set of positive integers, and $\mathbb{Z}$, the set of all integers, have the same cardinality. (CARD($\mathbb{Z}$) = CARD($\mathbb{Z}^+$))

Consider the following ways to describe the numbers in these sets:

$\mathbb{Z}^+ \xrightarrow{f} \mathbb{Z}$

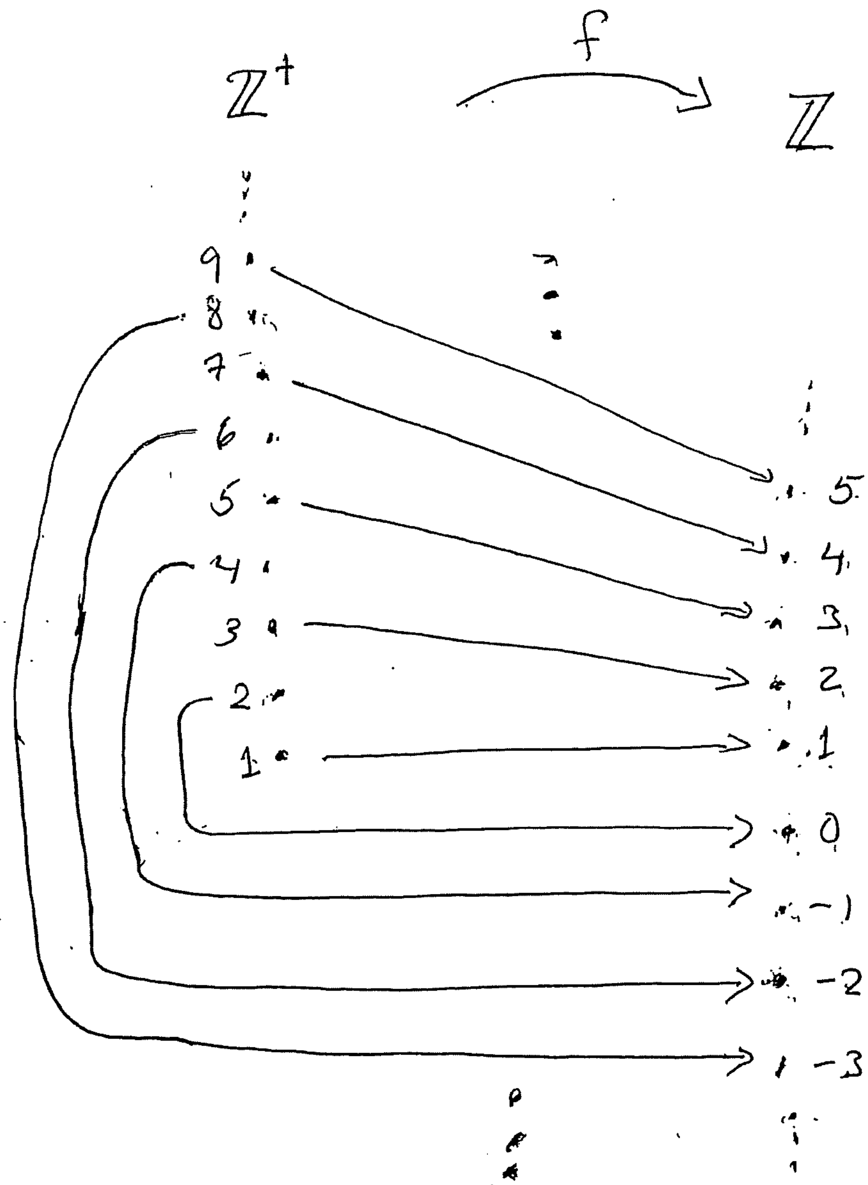
$\mathbb{Z}^+$		$\mathbb{Z}$	
⋮	⋮	⋮	⋮
7	= The 4th odd #		
6	= The 3rd EVEN #		
5	= The 3rd odd #	4	= The 4th pos. integer
4	= The 2nd EVEN #	3	= The 3rd pos. integer
3	= The 2nd odd #	2	= The 2nd pos. integer
2	= The 1st EVEN #	1	= The 1st pos. integer
1	= The 1st odd #		(COUNTING DOWN ↓)
		0	= The 1st NON-POS. integer
		−1	= The 2nd NON-POS. integer
		−2	= The 3rd NON-POS. integer
		−3	= The 4th non-pos. integer
		⋮	⋮

Define $f: \mathbb{Z}^+ \to \mathbb{Z}$ as follows:

For all $n \in \mathbb{Z}^+$,

if n is the k^{th} odd #, then $f(n)$ = the k^{th} pos. integer

and if n is the k^{th} even #, then $f(n)$ = the k^{th} non-pos. integer



In Formula Terms:

For every $n \in \mathbb{Z}^+$, $f(n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd.} \\ 1 - \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$

Function f is a one-to-one correspondence and $f : \mathbb{Z}^+ \to \mathbb{Z}$.

Thus, $\mathbb{Z}^+$ and $\mathbb{Z}$ have the SAME CARDINALITY.

PAGES 4 and 5 from "NOTES ON CARDINALITY"

Example 1: We show that the sets $\mathbb{Z}$ and $2\mathbb{Z}$ have the same cardinality, where $2\mathbb{Z}$ is the set $2\mathbb{Z} = \{ n \in \mathbb{Z} \mid n = 2k$ for some integer $k \} = \{$ all EVEN integers $\}$.

Proof: Define the function $f: \mathbb{Z} \to 2\mathbb{Z}$ as follows: For all $k \in \mathbb{Z}$, $f(k) = 2k$.

Then, define the function $g: 2\mathbb{Z} \to \mathbb{Z}$ as follows:

For all $n \in 2\mathbb{Z}$, $g(n) = \frac{1}{2}n$.

[Note: $\frac{1}{2}n \in \mathbb{Z}$: Since $n \in 2\mathbb{Z}$, $n = 2k$ for some integer k, so $\frac{1}{2}n = k \in \mathbb{Z}$.]

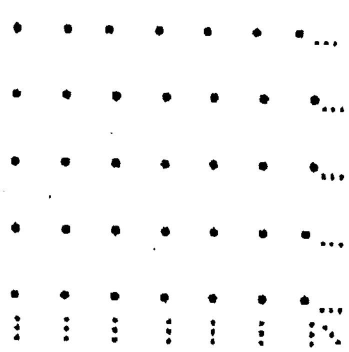
For all $k \in \mathbb{Z}$, $g \circ f(k) = g(f(k)) = \frac{1}{2}f(k) = \frac{1}{2}(2k) = k$.

For all $n \in 2\mathbb{Z}$, $f \circ g(n) = f(g(n)) = 2(g(n)) = 2(\frac{1}{2}n) = n$.

Therefore, by Theorem (NIB) 10, f is a one-to-one correspondence.

$\therefore \mathbb{Z}$ and $2\mathbb{Z}$ have the same cardinality. QED

Definition: A lattice is a finite or infinite collection of discrete points of the plane arranged in rows and columns. A standard example is the collection of points (x , y) in the Cartesian Coordinate Plane such that both coordinates x and y are integers. Even a single row or a single column from this collection is a lattice also.

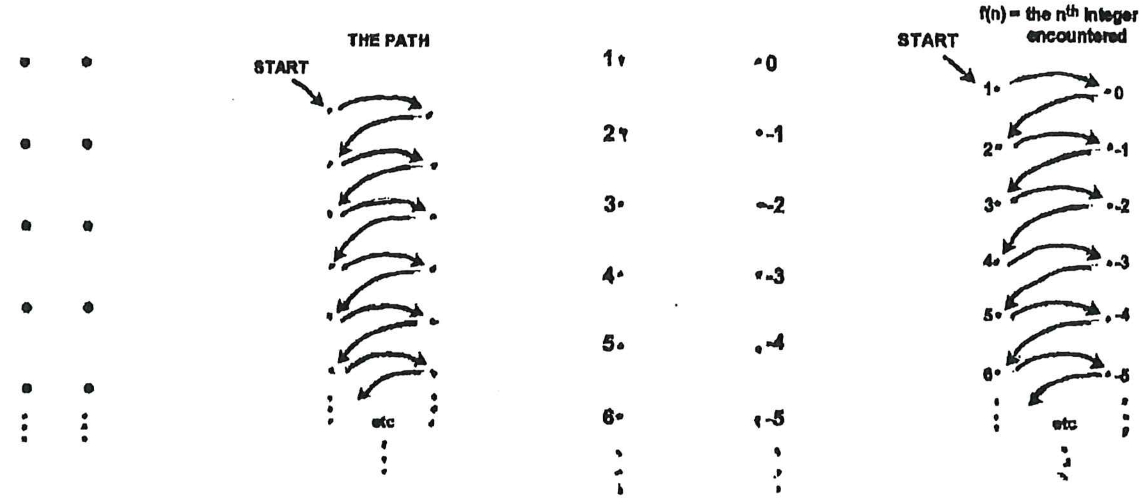


A LATTICE QUADRANT

LATTICES

Defining a Function $f: \mathbb{Z}^+ \to$ A SET in terms of a Path through a Lattice

We use a method of defining a function that is unconventional, but still it is a valid method for defining a function. We define a function in terms of a **path** through a lattice. Consider the first lattice below. We define a path through the lattice by indicating where the path begins and a systematic way the path proceeds so that every point in the lattice is eventually visited along the path exactly once. (See the second lattice below. Note that, when each point is visited, only a finite number of points have been visited previously!



We will define a function f from $\mathbb{Z}^+$ to $\mathbb{Z}$ using this unconventional method. First, we place all of the elements of the **co-domain** set on the points of the lattice which we have chosen to use. You can think of the placement process as a **labeling** of the points with elements of the co-domain of the function f. Here, $f: \mathbb{Z}^+ \to \mathbb{Z}$, so we systematically place each integer from $\mathbb{Z}$ on some point in the lattice. One such placement scheme is shown on the lattice above.

The Defining of the Function:

Place the integers in $\mathbb{Z}$ on the lattice points as shown above and then traverse the lattice along THE PATH indicated.

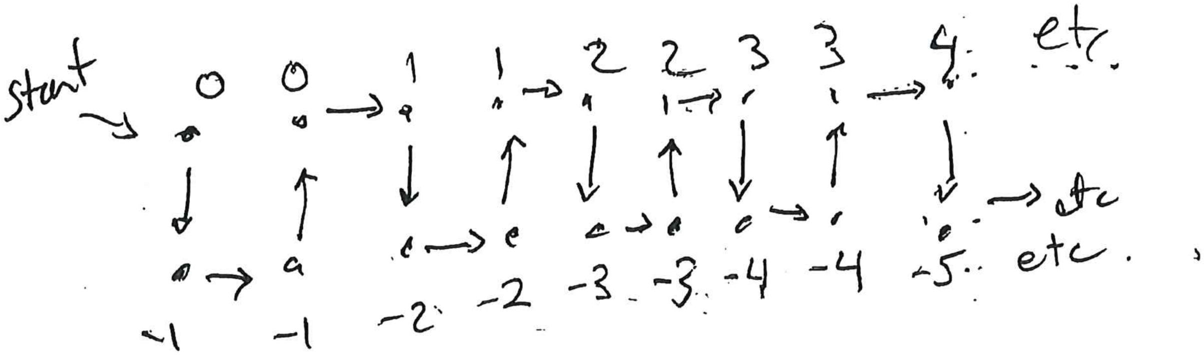
Define function $f: \mathbb{Z}^+ \to \mathbb{Z}$ as follows:

For all $n \in \mathbb{Z}^+$, $f(n)$ = the n^{th} integer in $\mathbb{Z}$ that is encountered along THE PATH through the lattice as shown above.

Thus, $f(1) = 1$, $f(2) = 0$, $f(3) = 2$, $f(4) = -1$, $f(5) = 3$, $f(6) = -2$, etc.

In this manner, function f is well-defined. For each positive integer n, the function value f(n) has been defined and can actually be known if we just travel down the path far enough.

Another EXAMPLE:



Define $g : \mathbb{Z}^+ \to \mathbb{Z}$ as follows:

For every $n \in \mathbb{Z}^+$, $g(n)$ is the n^{th} integer encountered along the path.

$g(1) = 0$, $g(2) = -1$, $g(3) = -1$, $g(4) = 0$, $g(5) = 1$, ...

The function g is onto, but g is not one-to-one,

e.g., $g(1) = 0$ and $g(4) = 0$, but $1 \neq 4$.

Using the same path and labelling:

Define $h : \mathbb{Z}^+ \to \mathbb{Z}$ as follows:

For every $n \in \mathbb{Z}^+$, $h(n)$ is the n^{th} integer that is encountered for the first time along the path

OR $h(n)$ = the n^{th} newly-encountered integer along the path

$h(1) = 0$, $h(2) = -1$, $h(3) = 1$, $h(4) = -2$, $h(5) = 2$,

$h(6) = -3$, . . .

Place representations of positive rational numbers on the lattice quadrant in this manner:

Defining $f: \mathbb{Z}^+ \to \mathbb{Q}^+$

$$
\begin{array}{lccccccccc}
\frac{1}{n}\text{'s} & \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \cdots \\
\frac{2}{n}\text{'s} & \frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \frac{2}{6} & \frac{2}{7} & \frac{2}{8} & \cdots \\
\frac{3}{n}\text{'s} & \frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \cdot & \cdot & \cdot & \cdot & \cdots \\
 & \frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{3} & \cdot & \cdot & \cdot & \cdot & \cdots \\
 & \frac{5}{1} & \frac{5}{2} & \frac{5}{3} & \frac{5}{4} & \cdot & \cdot & \cdot & \cdot & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

12)

Theorem (NIB) 13: $\mathbb{Q}^+$, the set of positive rational numbers, is countably infinite.

Proof: Clearly, $\mathbb{Q}^+$ is infinite since $\mathbb{Q}^+$ contains the set of positive integers.

To define a one-to-one correspondence from $\mathbb{Z}^+$ to $\mathbb{Q}^+$,
place the *representations* of the positive rational numbers on a lattice quadrant, as shown in Example 7.4.4 on page 338, Figure 7.4.3, as also shown below:

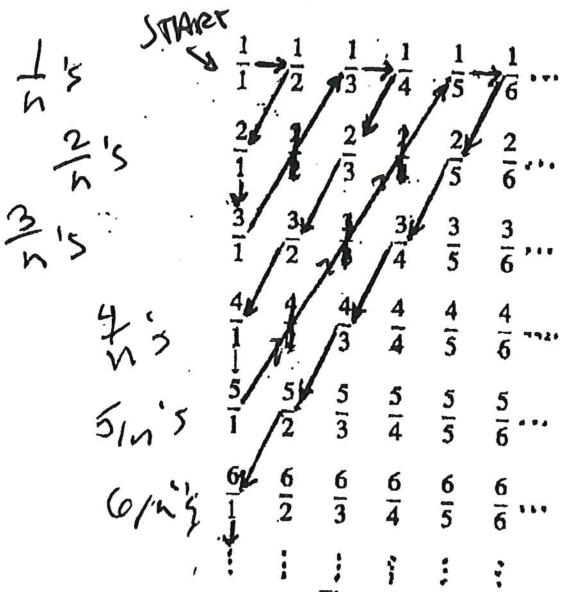


Figure 7.4.3

Traverse the lattice along the path indicated above.

Define $f: \mathbb{Z}^+ \to \mathbb{Q}^+$ as follows: For all $n \in \mathbb{Z}^+$,

$f(n)$ = the n^{th} **newly encountered** positive rational number, encountered along the path through the lattice indicated above.

Function f is one-to-one because previously-encountered rational numbers are skipped over.
Function f is onto because every positive rational number eventually is newly-encountered along the path.

Therefore, f is a one-to-one correspondence from $\mathbb{Z}^+$ to $\mathbb{Q}^+$.

$\therefore$ $\mathbb{Q}^+$ is countably infinite. QED

Note: This also proves that $\mathbb{Q}^+$ is a countable set, by definition of "countable".